

# NON-ZERO COEFFICIENTS OF HALF-INTEGRAL WEIGHT MODULAR FORMS MOD $\ell$

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**ABSTRACT.** We obtain new lower bounds for the number of Fourier coefficients of a weakly holomorphic modular form of half-integral weight not divisible by some prime  $\ell$ . Among the applications of this we show that there are  $\gg \sqrt{X}/\log \log X$  integers  $n \leq X$  for which the partition function  $p(n)$  is not divisible by  $\ell$ , and that there are  $\gg \sqrt{X}/\log \log X$  values of  $n \leq X$  for which  $c(n)$ , the  $n$ th Fourier coefficient of the  $j$ -invariant, is odd.

## 1. INTRODUCTION

Let  $K$  be a number field and  $\mathcal{O}$  its ring of integers. Let  $\ell$  be a rational prime and let  $\lambda$  be a maximal ideal of  $\mathcal{O}$  above  $\ell$ . We denote by  $\mathbb{F}$  the residue field  $\mathcal{O}/\lambda$ , a finite extension of  $\mathbb{F}_\ell$ . The reader will lose little by supposing that  $K = \mathbb{Q}$ ,  $\mathcal{O} = \mathbb{Z}$ ,  $\lambda = (\ell)$  and  $\mathbb{F} = \mathbb{F}_\ell$ ; our main applications use only this case.

**Theorem 1.** *Let  $f = \sum_{n=n_0}^{\infty} a_n q^n$  be a weakly holomorphic modular form<sup>1</sup> of weight  $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$  and level  $\Gamma_1(N)$ . Suppose that the coefficients  $a_n$  lie in the ring  $\mathcal{O}$ . If  $\ell \geq 3$ , we assume that  $f \not\equiv 0 \pmod{\lambda}$ , and for  $\ell = 2$  we assume that  $f \pmod{\lambda}$  is not a constant. Then*

$$\#\{n \leq X, a_n \not\equiv 0 \pmod{\lambda}\} \gg \frac{\sqrt{X}}{\log \log X}.$$

Here are some sample applications of Theorem 1.

**Example 1.** Take  $f = \eta_1(z)^{-1}$  with  $\eta_1(z) = \eta(24z)$  (Dedekind's eta function), so that  $f$  is a weakly holomorphic modular form of weight  $-1/2$  and level  $\Gamma_0(576)$ . The Fourier expansion of  $f$  is

$$f(q) = q^{-1} \prod_{n=1}^{\infty} (1 - q^{24n})^{-1} = \sum_{n=0}^{\infty} p(n) q^{24n-1}$$

where  $p(n)$  is the *partition function* (cf. [15, Theorem 1.60, Corollary 1.62 and Theorem 5.3] for a proof of these well-known facts). Applying the theorem to  $f$ , we conclude that

$$(1) \quad \#\{n \leq X, p(n) \not\equiv 0 \pmod{\ell}\} \gg \frac{\sqrt{X}}{\log \log X}.$$

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<sup>1</sup> *Weakly holomorphic* allows for polar singularities at the cusps; for this and other basic definitions, we refer the reader to [15, Chapter 1].

This improves, by a factor of about  $(\log X)^{\frac{3}{4}}$ , earlier results of Ahlgren [1], Chen [8] and Dai & Fang [9] for odd  $\ell$ . In the case  $\ell = 2$ , (1) improves upon previous results (established by somewhat different methods than for  $\ell$  odd) by a factor of about  $(\log X)^{\frac{7}{8}}$ ; see [4, 12, 13, 14].

**Example 2.** Theorem 1 applies in particular when  $f$  is a *holomorphic* cusp form of half-integral weight  $k$  (which must then be positive). In this case, it improves on the main theorem of [7] (itself an improvement of [16]) which proves the slightly weaker estimate  $\#\{n \leq X, a_n \not\equiv 0 \pmod{\ell}\} \gg \sqrt{X}/\log X$  under the supplementary assumption that the coefficients  $a_n \pmod{\lambda}$  are not supported in a finite union of sequences of the form  $(cn^2)_{n \in \mathbb{N}}$ . We remark that in [7, 16] the Shimura correspondence between holomorphic half-integral weight modular forms and integral weight modular forms plays a crucial role, whereas our proof of Theorem 1 does not involve the Shimura correspondence.

**Example 3.** When  $\ell = 2$ , our theorem applies as well to weakly holomorphic modular forms of *integral* weight, since those forms are congruent modulo  $\lambda$  to forms of *half-integral* weight (see Lemma 3 below). In particular, for the modular invariant  $j(q) = \sum_{n=-1}^{\infty} c(n)q^n$ , which is of weight 0 and level  $\mathrm{SL}_2(\mathbb{Z})$ , we obtain that

$$(2) \quad \#\{n \leq X, c(n) \text{ is odd}\} \gg \frac{\sqrt{X}}{\log \log X}.$$

This improves upon recent results in [2, 17] obtained by different methods.

For completeness, we remark that [5] establishes, for  $\ell \geq 3$ , an asymptotic for the number of non-zero coefficients  $(\bmod \ell)$  of holomorphic modular forms, and [4] establishes such an asymptotic for  $\ell = 2$  and holomorphic forms of level 1. The situation for *weakly* holomorphic forms of integral weight remains mysterious, and (for example) we do not have lower bounds for the number of  $c(n) \not\equiv 0 \pmod{\ell}$  for primes  $3 \leq \ell \leq 11$ .

The proof of Theorem 1 uses the standard idea of multiplying  $f$  by a suitable lacunary holomorphic cusp form  $g$  of half-integral weight, to obtain an holomorphic cusp form  $h = fg$  of integral weight. The panoply of results stemming from the existence of Galois representations associated to integral weight holomorphic eigenforms may then be used to study the coefficients of  $h$ . Finding a suitable form  $g$  is easy when  $\ell > 2$ , and somewhat less so in the case  $\ell = 2$ . Our improvement over previous work comes from analyzing more carefully the implications for non-vanishing coefficients of the equality  $h = fg$ , which leads to a problem in analytic number theory/additive combinatorics.

**Theorem 2.** *Let  $u \geq 1$  be a fixed natural number, and let  $X$  be large. For any subset  $\mathcal{A} \subset \{1, \dots, X\}$  the number of primes  $p$  such that  $pu \leq X$  and  $pu$  may be written as  $a + m^2$  for some  $a \in \mathcal{A}$  and some integer  $m$  is*

$$\ll \frac{\sqrt{X}}{\log X} \left( |\mathcal{A}| \log \log X + |\mathcal{A}|^{\frac{1}{2}} X^{\frac{1}{4}} \right).$$

Our interest in Theorem 2 is in the situation where a positive proportion of the primes  $p$  are known to be of the form  $a + m^2$ , when it follows that  $|\mathcal{A}|$  must have  $\gg \sqrt{X}/\log \log X$  elements. This statement is optimal, as we shall show in section 4 by constructing an example of a set  $\mathcal{A}$  with  $|\mathcal{A}| \asymp \sqrt{X}/\log \log X$ , and with a positive proportion of primes below  $X$  being of the form  $a + m^2$ .

Theorem 1, on the other hand, is almost certainly not optimal. For any weakly holomorphic form  $f(q) = \sum a_n q^n$  of half-integral weight, one might expect

$$\#\{n \leq X, a_n \not\equiv 0 \pmod{\lambda}\} \gg \sqrt{X},$$

and this bound is attained for  $\eta_1(q)$  (see (3) below). Theorem 1 comes close to this estimate. For most forms  $f$  of half-integral weight however (specifically for  $f(q) = \eta_1^{-1}(q) = \sum_n p(n)q^{24n-1}$ , and perhaps for all forms that are not congruent mod  $\lambda$  to a one-variable theta series), it is expected that  $f \pmod{\lambda}$  is not *lacunary*, which is to say that  $\#\{n \leq X, a_n \not\equiv 0 \pmod{\lambda}\} \gg X$ .

## 2. DEDUCTION OF THEOREM 1 FROM THEOREM 2

**2.1. A preliminary lemma.** Let  $M_k(\Gamma_1(N), \mathcal{O})$  be the  $\mathcal{O}$ -module of holomorphic modular forms of integral weight  $k \geq 0$ , level  $\Gamma_1(N)$ , and coefficients in  $\mathcal{O}$ . Let  $\mathcal{O}_\lambda$  be the completion of  $\mathcal{O}$  at the place defined by the ideal  $\lambda$ , and set  $M_k(\Gamma_1(N), \mathcal{O}_\lambda) = M_k(\Gamma_1(N), \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}_\lambda$ . Let  $A$  be the closure of the  $\mathcal{O}_\lambda$ -subalgebra of  $\text{End}_{\mathcal{O}_\lambda}(M_k(\Gamma_1(N), \mathcal{O}_\lambda))$  generated by the Hecke operators  $T_n$  for  $n$  running among integers relatively primes to  $N\ell$ . Denote by  $G_{\mathbb{Q}, N\ell}$  the Galois group of the maximal extension of  $\mathbb{Q}$  unramified outside  $N\ell$ , and by  $\text{Frob}_p$ , for  $p$  a prime not dividing  $N\ell$ , the Frobenius element of  $p$  in  $G_{\mathbb{Q}, N\ell}$ , well-defined up to conjugation.

**Lemma 1.** *There exists a unique continuous map  $t : G_{\mathbb{Q}, N\ell} \rightarrow A$  which is central and satisfies  $t(\text{Frob}_p) = T_p$  for every prime  $p$  not dividing  $N\ell$ . This map also satisfies  $t(1) = 2$ .*

*Proof.* This follows from a well-known argument of Wiles based on the existence of Galois representations attached to eigenforms due to Deligne; see, for example, [3, Thm 1.8.5] for a detailed proof.  $\square$

**2.2. The case  $\ell > 2$ .** We begin with a lemma.

**Lemma 2.** *Assume that  $\ell$  is odd. Let  $k \in \mathbb{N}$  and  $h(q) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_1(N), \mathcal{O})$ . Let  $u \geq 1$  be an integer such that  $a_u \not\equiv 0 \pmod{\lambda}$ . There is a positive density set of primes  $\mathcal{P}$  such that  $a_{up} \not\equiv 0 \pmod{\lambda}$  for every  $p \in \mathcal{P}$ .*

*Proof.* With  $t$  as in Lemma 1, the map from  $G_{\mathbb{Q}, N\ell}$  to  $\mathbb{F}$  sending  $g$  to  $a_u(t(g)h) \pmod{\lambda}$  is a continuous map. Thus there exists an open neighborhood  $U$  of 1 in  $G_{\mathbb{Q}, N\ell}$  such that  $g \mapsto a_u(t(g)h) \pmod{\lambda}$  is constant on  $U$ . Let  $\mathcal{P}$  be the set of primes not dividing  $N\ell u$  such that  $\text{Frob}_p \in U$ . By Chebotarev,  $\mathcal{P}$  has positive density. Further for  $p \in \mathcal{P}$  we have

$$a_{up}(h) = a_u(T_p h) = a_u(t(\text{Frob}_p)h) = a_u(t(1)h) = 2a_u(h) \not\equiv 0 \pmod{\lambda},$$

since  $\ell \neq 2$  and  $a_u(h) \not\equiv 0 \pmod{\lambda}$ .  $\square$

We can now deduce Theorem 1 from Theorem 2. Let  $f = \sum_{n \geq n_0} a_n q^n$  be a weakly holomorphic modular form of half-integral weight, level  $\Gamma_1(N)$ , and coefficients in  $\mathcal{O}_\lambda$ . Let  $\eta(z)$  be the usual Dedekind's eta function and set  $\eta_1(z) = \eta(24z)$  so that (see [15])

$$(3) \quad \eta_1(q) = q \prod_n (1 - q^{24n}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2}$$

is a holomorphic cuspidal modular form of weight  $1/2$ . Let  $m$  be an even integer such that  $\ell^m$  is larger than the order of any pole of  $f$ . Then  $h = f\eta_1^{\ell^m}$  is a holomorphic cuspidal modular form of integral weight  $k + \ell^m/2$ . Since  $f \pmod{\lambda}$  and  $\eta_1 \pmod{\lambda}$  are non-zero, the power series  $h \pmod{\lambda} \in \mathbb{F}[[q]]$  is also non-zero, and indeed  $h \pmod{\lambda}$  is not a constant (because  $h$  is cuspidal, and a cuspidal constant form must be 0).

Let  $\mathcal{A} = \{n, a_n = a_n(f) \not\equiv 0 \pmod{\lambda}\}$ . Note that from (3)

$$\eta_1(q)^{\ell^m} = \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \right)^{\ell^m} \equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{\ell^m(6n+1)^2} \pmod{\lambda},$$

so that the Fourier coefficients of  $\eta_1^{\ell^m}$  are non-zero  $\pmod{\lambda}$  only on squares. Thus if  $n$  is such that  $a_n(h) \not\equiv 0 \pmod{\lambda}$ , then  $n$  must be of the form  $a + m^2$  for some  $a \in \mathcal{A}$  and some integer  $m$ . Now, by Lemma 2, the set of  $n$  such that  $a_n(h) \not\equiv 0 \pmod{\lambda}$  contains a set of the form  $u\mathcal{P}$ , for some fixed natural number  $u$  and a set of primes  $\mathcal{P}$  of positive density. For large  $X$ , it follows from Theorem 2 that the number of primes  $p$  with  $up \leq X$  and  $up$  of the form  $a + m^2$  with  $a \in \mathcal{A}$  is  $\ll |\mathcal{A} \cap [1, X]| \sqrt{X} (\log \log X) / \log X + |\mathcal{A} \cap [1, X]|^{\frac{1}{2}} X^{\frac{3}{4}} / \log X$ . It follows that  $|\mathcal{A} \cap [1, X]| \gg \sqrt{X} / \log \log X$ , proving Theorem 1.

**2.3. The case  $\ell = 2$ .** This case needs a little more care, and we begin by recalling a well-known result that (for  $\ell = 2$ ) modular forms of integer weights are congruent to modular forms of half-integer weight.

**Lemma 3.** *Assume that  $\ell = 2$ . For every weakly holomorphic modular form  $f$  of weight  $k$  and level  $\Gamma_1(N)$  with coefficients in  $\mathcal{O}$  there exists a weakly holomorphic modular form  $f'$  of weight  $k + 1/2$ , some level  $\Gamma_1(N')$ , with coefficients in  $\mathcal{O}$ , such that  $f \equiv f' \pmod{\lambda}$ .*

*Proof.* Recall that (see, for example, [15, Prop 1.4]) the theta series  $\theta_0(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + \sum_{n=1}^{\infty} 2q^{n^2}$  is a holomorphic modular form of weight  $1/2$ , level  $\Gamma_0(4)$ , coefficients in  $\mathbb{Z}$ . Now take  $f' = f\theta_0$ .  $\square$

**Lemma 4.** *Let  $n_0$  be a non-zero integer, and let  $N$  be a positive integer. There are only finitely many natural numbers  $m$  such that  $2^m + n_0$  equals  $uy^2$  for some square-free divisor  $u$  of  $2N$ .*

*Proof.* Write  $m = 3m_0 + r$  with  $r = 0, 1$ , or  $2$ , and set  $x = 2^{m_0}$ . The equation  $2^m + n_0 = uy^2$  becomes  $2^r x^3 + n_0 = uy^2$ , which for a given  $r$  and  $u$  may be viewed as an elliptic curve (since  $n_0 \neq 0$ ). By Siegel's theorem there are only finitely many integer points  $(x, y)$  on this elliptic curve. Since there are only three possible values for  $r$ , and finitely many possibilities for  $u$  (being a square-free divisor of  $2N$ ), the lemma follows.  $\square$

**Lemma 5.** *Let  $k \in \mathbb{N}$  and  $h(q) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_1(N), \mathcal{O})$ . Assume that there exists an integer  $n \geq 1$  and a prime  $p_0$  not dividing  $2N$  such that  $\text{ord}_{p_0} n$  is odd and  $a_n \not\equiv 0 \pmod{\lambda}$ . Then there exists an integer  $u \geq 1$  and a set of primes  $\mathcal{P}$  of positive density such that  $a_{up} \not\equiv 0 \pmod{\lambda}$  for every  $p \in \mathcal{P}$ .*

*Proof.* We recall that if  $h = \sum_{n=0}^{\infty} a_n q^n$  is a modular form for  $\Gamma_1(N)$ , and if  $p$  is a prime not dividing  $N$ , the  $m$ -th coefficient of the form  $T_p h$  is

$$(4) \quad a_m(T_p h) = a_{mp}(h) + p^{k-1} a_{m/p}(\langle p \rangle h),$$

where  $\langle p \rangle$  is the diamond operator, and with the convention that  $a_{m/p}(-) = 0$  when  $p \nmid m$ .

We claim that *if  $p$  is a prime not dividing  $2N$  and if  $T_p h \pmod{\lambda}$  is a constant, then  $a_n(h) = 0$  if  $\text{ord}_p(n)$  is odd.* We prove that claim, for all  $h$  such that  $T_p h \pmod{\lambda}$  is constant, by induction over the odd number  $\text{ord}_p(n)$ . If  $\text{ord}_p(n) = 1$ , applying (4) to the form  $h$  and the integer  $m = n/p$  and reducing mod  $\lambda$  gives (using that  $p \equiv -1 \equiv 1 \pmod{\lambda}$ ):

$$a_n(h) \equiv a_{n/p^2}(\langle p \rangle h) = 0 \pmod{\lambda}.$$

For a general  $n$  with  $\text{ord}_p(n)$  odd, we get similarly

$$a_n(h) \equiv a_{n/p^2}(\langle p \rangle h) \pmod{\lambda}.$$

By the induction hypothesis applied to the form  $\langle p \rangle h$  (which also satisfies  $T_p(\langle p \rangle h) \pmod{\lambda}$  constant since the diamond operator  $\langle p \rangle$  commutes with  $T_p$  and stabilizes the subspace of constants), we get  $a_n(h) \equiv 0 \pmod{\lambda}$  which completes the induction step.

By the hypothesis of the Lemma, it follows that  $T_{p_0} h \pmod{\lambda}$  is not a constant, that is to say there exists  $u \geq 1$  such that  $a_u(T_{p_0} h) \not\equiv 0 \pmod{\lambda}$ , or equivalently, with  $t$  as in Lemma 1,  $a_u(t(\text{Frob}_{p_0})h) \not\equiv 0 \pmod{\lambda}$ . By continuity of  $t$ , there exists an open set  $U$  in  $G_{\mathbb{Q}, Np}$  such that for  $p$  a prime not dividing  $Nu$ , if  $\text{Frob}_p \in U$ , then

$$a_{up}(h) = a_u(T_p h) = a_u(t(\text{Frob}_p)h) \equiv a_u(t(\text{Frob}_{p_0})h) \not\equiv 0 \pmod{\lambda}.$$

The set  $\mathcal{P}$  of such primes  $p$  is a set of primes of positive density by Chebotarev.  $\square$

We are now ready to prove Theorem 1 in the case  $\ell = 2$  using Theorem 2. Let  $f$  be a weakly holomorphic modular form of half-integral weight with coefficients in  $\mathcal{O}_{\lambda}$ . By Lemma 3 we may assume that  $f$  has integral weight instead.

We consider the form  $h := f\eta_1^{2^m}$  for a suitable  $m$  that will be specified below. We observe that if  $m \geq 1$ ,  $\eta_1^{2^m}$  has integral weight, and so does  $h$ . Moreover, since  $\eta_1$  is cuspidal,  $h$  is also cuspidal holomorphic for  $m$  large enough.

Write  $f \equiv \sum_{n=n_0}^{\infty} a_n q^n \pmod{\lambda}$  with  $a_{n_0} \neq 0$ . The first term of  $h \pmod{\lambda}$  is  $a_{n_0} q^{2^m + n_0}$ . When  $n_0 \neq 0$ , Lemma 4 ensures that we can choose  $m$  even, large enough in the sense of the preceding paragraph, and such that there is a prime  $p_0$  not dividing  $2N$  such that  $\text{ord}_{p_0}(2^m + n_0)$  is odd. When  $n_0 = 0$ , let  $a_{n_1} q^{n_1}$  with  $n_1 > 0$ ,  $a_{n_1} \neq 0$  the term of smallest positive degree in  $f \pmod{\lambda}$  (such an  $n_1$  exists because we assume that  $f \pmod{\lambda}$  is not a constant). If  $m$  is such that  $2^m > n_1$ , then the form  $h$  has a term  $a_{n_1} q^{2^m + n_1}$ . Again by Lemma 4 we can find  $m$  large enough and such that there is a prime  $p_0$  not dividing  $2m$  such that  $\text{ord}_{p_0}(2^m + n_1)$  is odd.

So in both cases ( $n_0 \neq 0$  and  $n_0 = 0$ ) we have shown the existence of an integer  $m$  such that  $h = f\eta_1^{2^m}$  is a cuspidal holomorphic modular form of integral weight and such that, by Lemma 5, there is  $u \geq 1$  and a set of primes  $\mathcal{P}$  of positive density, with  $a_{up}(h) \not\equiv 0 \pmod{\lambda}$  for every  $p \in \mathcal{P}$ . The rest of the proof is now exactly as in the case  $\ell > 2$ .

## 3. PROOF OF THEOREM 2

Given a positive integer  $a$ , we let  $\chi_{-4a} = (\frac{-4a}{\cdot})$  denote the Kronecker symbol, which is a Dirichlet character  $(\bmod 4a)$ . Note that  $-4a$  is a discriminant, but it need not be a fundamental discriminant. We denote the associated (negative) fundamental discriminant by  $\tilde{a}$ , so that  $-4a = \tilde{a}a_2^2$  for a suitable natural number  $a_2$ .

**Lemma 6.** *Let  $u$  be a fixed natural number, and let  $X$  be large. For every integer  $1 \leq a \leq X$ ,*

$$\#\{p : up \leq X, p = a + m^2 \text{ for some } m \in \mathbb{Z}\} \ll \frac{\sqrt{X}}{\log X} \prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{-4a}(p)}{p}\right).$$

*Proof.* Consider the equivalent problem of estimating the number of  $m$  below  $\sqrt{X}$  such that  $a + m^2$  is of the form  $ur$  for a prime number  $r$ . We may restrict attention to  $r > X^{\frac{1}{4}}$ , since the smaller primes  $r$  contribute negligibly to the number of  $m$ . For each prime  $p \nmid 2u$  and  $p \leq X^{\frac{1}{4}}$  we see that  $m^2$  cannot be  $\equiv -a \pmod{p}$ , which means that  $1 + \chi_{-4a}(p)$  residue classes  $(\bmod p)$  are forbidden for  $m$ . Any standard upper bound sieve (for example, Brun's sieve or Selberg's sieve; or see Theorem 2.2 of [11]) then shows that the number of possible  $m \leq \sqrt{X}$  is

$$\ll \sqrt{X} \prod_{\substack{p \leq X^{\frac{1}{4}} \\ p \nmid 2u}} \left(1 - \frac{1 + \chi_{-4a}(p)}{p}\right) \ll \frac{\sqrt{X}}{\log X} \prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{-4a}(p)}{p}\right),$$

and the lemma follows.  $\square$

Call a fundamental discriminant  $d$  *good* if the corresponding Dirichlet  $L$ -function  $L(s, \chi_d)$  has no zeros in the region  $\{\sigma > 99/100, |t| \leq |d|\}$ , and call the discriminant  $d$  *bad* otherwise.

**Lemma 7.** *Suppose  $1 \leq a \leq X$  is an integer, and that the fundamental discriminant  $\tilde{a}$  corresponding to  $-4a$  is good. Then*

$$\prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{-4a}(p)}{p}\right) \ll \log \log X.$$

*Proof.* By [10, Lemma 2.1], for a good fundamental discriminant  $\tilde{a}$  one has

$$L\left(1 + \frac{1}{\log X}, \chi_{\tilde{a}}\right) \asymp \prod_{p < (\log |\tilde{a}|)^{100}} \left(1 - \frac{\chi_{\tilde{a}}(p)}{p}\right)^{-1}.$$

Further

$$\begin{aligned} \log L\left(1 + \frac{1}{\log X}, \chi_{\tilde{a}}\right) &= \sum_p \frac{\chi_{\tilde{a}}(p)}{p^{1+1/\log X}} + O(1) \\ &= \sum_{p \leq X^{\frac{1}{4}}} \frac{\chi_{\tilde{a}}(p)}{p} + O\left(\sum_{p \leq X^{\frac{1}{4}}} \frac{1 - p^{-1/\log X}}{p} + \sum_{p > X^{\frac{1}{4}}} \frac{1}{p^{1+1/\log X}} + 1\right). \end{aligned}$$

Using  $1 - p^{-1/\log X} = O(\frac{\log p}{\log X})$  for  $p \leq X^{\frac{1}{4}}$ , the first error term above is seen to be  $O(1)$ , and partial summation shows that the second term is also  $O(1)$ . Therefore

$$(5) \quad \prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{\tilde{a}}(p)}{p}\right) \asymp L(1 + 1/\log X, \chi_{\tilde{a}})^{-1} \asymp \prod_{p \leq (\log |\tilde{a}|)^{100}} \left(1 - \frac{\chi_{\tilde{a}}(p)}{p}\right).$$

Now write  $-4a = \tilde{a}a_2^2$  for some positive integer  $a_2 \leq \sqrt{X}$ . Then

$$\prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{-4a}(p)}{p}\right) = \prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{\tilde{a}}(p)}{p}\right) \prod_{\substack{p \leq X^{\frac{1}{4}} \\ p|a_2}} \left(1 - \frac{\chi_{\tilde{a}}(p)}{p}\right)^{-1},$$

and using (5) this is

$$(6) \quad \asymp \prod_{\substack{p \leq (\log |\tilde{a}|)^{100} \\ p \nmid a_2}} \left(1 - \frac{\chi_{\tilde{a}}(p)}{p}\right) \prod_{\substack{X^{\frac{1}{4}} \geq p \geq (\log |\tilde{a}|)^{100} \\ p|a_2}} \left(1 - \frac{\chi_{\tilde{a}}(p)}{p}\right)^{-1}.$$

The first product in (6) is clearly at most

$$\prod_{p \leq (\log |\tilde{a}|)^{100}} \left(1 + \frac{1}{p}\right) \ll \log \log |\tilde{a}|.$$

As for the second product in (6), this is

$$\leq \prod_{\substack{X^{\frac{1}{4}} \geq p \geq (\log |\tilde{a}|)^{100} \\ p|a_2}} \left(1 - \frac{1}{p}\right)^{-1} \leq \prod_{(\log \tilde{a})^{100} \leq p \leq (\log \tilde{a})^{100} + (\log X)^2} \left(1 - \frac{1}{p}\right)^{-1},$$

since  $a_2$  has at most  $\log X$  prime factors, and the product is largest if these prime factors are the first  $\leq \log X$  primes all larger than  $(\log \tilde{a})^{100}$ . This quantity is easily seen to be  $\ll \max(1, \frac{\log \log X}{\log \log |\tilde{a}|})$ , proving the lemma.  $\square$

Applying Lemmas 6 and 7 we see that the number of primes  $p$  with  $up \leq X$  and  $p$  of the form  $a + m^2$  with  $a \in \mathcal{A}$  coming from a *good* associated fundamental discriminant  $\tilde{a}$  is bounded by

$$\sum_{\substack{a \in \mathcal{A} \\ \tilde{a} \text{ good}}} \frac{\sqrt{X}}{\log X} \prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{-4a}(p)}{p}\right) \ll |\mathcal{A}| \frac{\sqrt{X}}{\log X} \log \log X.$$

It remains to bound the number of primes arising from *bad* fundamental discriminants  $\tilde{a}$ . Note that, with  $-4a = \tilde{a}a_2^2$ ,

$$\begin{aligned} \prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{-4a}(p)}{p}\right) &\ll \prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{\tilde{a}}(p)}{p}\right) \prod_{p|a_2} \left(1 - \frac{1}{p}\right)^{-1} \\ &\ll \frac{a_2}{\phi(a_2)} L(1 + 1/\log X, \chi_{\tilde{a}})^{-1} \ll |\tilde{a}|^\epsilon \frac{a_2}{\phi(a_2)}, \end{aligned}$$

where the final estimate follows by an obvious modification to Siegel's theorem which gives  $L(1 + 1/\log X, \chi_{\tilde{a}}) \gg |\tilde{a}|^{-\epsilon}$ . Thus the number of primes arising from bad fundamental discriminants is

$$\ll \frac{\sqrt{X}}{\log X} \sum_{\substack{a \leq X \\ \tilde{a} \text{ bad}}} \prod_{p \leq X^{\frac{1}{4}}} \left(1 - \frac{\chi_{-4a}(p)}{p}\right) \ll \frac{\sqrt{X}}{\log X} \sum_{\substack{|\tilde{a}| \leq X \\ \tilde{a} \text{ bad}}} |\tilde{a}|^\epsilon \sum_{\substack{-4a = \tilde{a}a_2^2 \\ a \in \mathcal{A}}} \frac{a_2}{\phi(a_2)}.$$

For a given  $\tilde{a}$  we may bound the sum over  $a_2$  above using Cauchy-Schwarz; thus

$$\sum_{\substack{-4a = \tilde{a}a_2^2 \\ a \in \mathcal{A}}} \frac{a_2}{\phi(a_2)} \leq \left( \sum_{\substack{-4a = \tilde{a}a_2^2 \\ a \in \mathcal{A}}} 1 \right)^{\frac{1}{2}} \left( \sum_{a_2 \leq \sqrt{X/|\tilde{a}|}} \left( \frac{a_2}{\phi(a_2)} \right)^2 \right)^{\frac{1}{2}} \ll \sqrt{|\mathcal{A}|} \frac{X^{\frac{1}{4}}}{|\tilde{a}|^{\frac{1}{4}}}.$$

We conclude that the number of primes  $p$  arising from bad fundamental discriminants is

$$(7) \quad \ll \frac{\sqrt{X}}{\log X} |\mathcal{A}|^{\frac{1}{2}} X^{\frac{1}{4}} \sum_{\substack{|\tilde{a}| \leq X \\ \tilde{a} \text{ bad}}} \frac{|\tilde{a}|^\epsilon}{|\tilde{a}|^{\frac{1}{4}}} \ll \frac{|\mathcal{A}|^{\frac{1}{2}} X^{\frac{3}{4}}}{\log X} \sum_{\substack{|\tilde{a}| \leq X \\ \tilde{a} \text{ bad}}} |\tilde{a}|^{-\frac{1}{6}},$$

upon choosing  $\epsilon = 1/12$ . At this stage we note that bad fundamental discriminants are rare by a standard zero density result (see for example [6, Theorem 20]): thus there are at most  $\ll Y^{1/10}$  bad fundamental discriminants  $d$  with  $Y \leq |d| \leq 2Y$ . Therefore the sum over bad  $\tilde{a}$  in (7) converges, and we conclude that the quantity in (7) is  $\ll |\mathcal{A}|^{\frac{1}{2}} X^{\frac{3}{4}} / \log X$ . This completes the proof of Theorem 2.

#### 4. OPTIMALITY OF THEOREM 2

In this section, we show the existence of a subset  $\mathcal{A}$  of  $[1, X]$  with  $|\mathcal{A}| \asymp \sqrt{X} / \log \log X$ , and such that a positive proportion of the primes below  $X$  may be written as  $a + m^2$  with  $a \in \mathcal{A}$  and  $m \in \mathbb{Z}$ . Since this is only an example to show the optimality of Theorem 2, we shall be content with sketching the proof.

Set  $Z = \exp((\log X)^{\frac{1}{10}})$ . Note that  $\log \log Z \asymp \log \log X$ . Let  $\mathcal{D}$  be a set of about  $\sqrt{Z} / \log \log X$  odd square-free numbers  $d$  with  $Z \leq d \leq 2Z$  and such that  $L(1, \chi_{-4d}) \asymp 1 / \log \log X$ . Then our set  $\mathcal{A}$  will consist of all numbers of the form  $dk^2$  with  $d \in \mathcal{D}$  and  $k \leq \sqrt{X/2Z}$ . By construction the set  $\mathcal{A}$  has  $\asymp \sqrt{X} / \log \log X$  elements.

Arguing using a classical zero-free region for class group  $L$ -functions, we may see that for any  $d \in \mathcal{D}$  the number of primes up to  $X/2$  of the form  $dk^2 + b^2$  with  $b, k \in \mathbb{N}$  is

$$\gg \frac{\pi(X)}{h(-4d)} \asymp \frac{X}{\sqrt{Z} \log X} \log \log X,$$

upon using the class number formula. Thus if  $r_{\mathcal{A}}(p)$  denotes the number of ways of writing  $p$  as  $a + b^2$  with  $a \in \mathcal{A}$  and  $b \in \mathbb{N}$ , it follows that

$$\sum_{p \leq X/2} r_{\mathcal{A}}(p) \gg \frac{X}{\log X}.$$

By similar methods, we may show that for  $d_1 \neq d_2 \in \mathcal{D}$ , the number of primes up to  $X/2$  that may be represented as  $d_1 k^2 + b^2$  and also as  $d_2 r^2 + s^2$  is at most

$$\ll \frac{\pi(X)}{h(-4d_1)h(-4d_2)} \asymp \frac{X}{Z \log X} (\log \log X)^2.$$

It follows that

$$\sum_{p \leq X/2} r_{\mathcal{A}}(p)^2 \ll \frac{X}{\log X}.$$

By Cauchy-Schwarz it follows that the number of  $p \leq X/2$  with  $r_{\mathcal{A}}(p) > 0$  is  $\gg X / \log X$ , as claimed.

In Theorem 2 we were interested in lower bounds for the size of a set  $\mathcal{A} \subset \{1, \dots, X\}$  such that  $\mathcal{A} + \mathcal{B} \supset \mathcal{C}$ , where  $\mathcal{B}, \mathcal{C} \subset \{1, \dots, X\}$  are given sets (in this case,  $\mathcal{B}$  is the set of squares, and  $\mathcal{C}$  is a set consisting of a positive proportion of the primes). One might say that  $\mathcal{A}$  is an *additive complement of  $\mathcal{B}$  relative to  $\mathcal{C}$* . In the case  $\mathcal{C} = \{1, \dots, X\}$  one recovers the usual notion of additive complement. To our knowledge the relative case has not been studied in any generality. A large number of questions suggest themselves.



## REFERENCES

- [1] S. Ahlgren, Non vanishing of the partition function modulo odd primes. *Mathematika* 46 (1999), 185–192
- [2] C. Alfes, Parity of the coefficients of Klein’s  $j$ -function. *Proc. Amer. Math. Soc.* 141 (1) (2013) 123–130.
- [3] J. Bellaïche, Eigenvarieties, families of Galois representations,  $p$ -adic  $L$ -functions. Notes available on [people.brandeis.edu/~jbellaic](http://people.brandeis.edu/~jbellaic)
- [4] J. Bellaïche & J.-L. Nicolas, Parité des coefficients de formes modulaires. *Ramanujan J.* 40 (2016), no. 1, 1–44.
- [5] J. Bellaïche & K. Soundararajan, The number of nonzero coefficients of modular forms (mod  $p$ ). *Algebra Number Theory* 9 (2015), no. 8, 1825–1856.
- [6] E. Bombieri, Le grand crible dans la théorie analytique des nombres, *Astérisque* 18 (1987/1974).
- [7] J. H. Bruinier & K. Ono, Coefficients of half-integral weights modular forms. *Journal of Number Theory* 99 (2003), 164–179
- [8] S.-C. Chen, Distribution of the coefficients of modular forms and the partition function. *Arch. Math.* 98 (2012), no. 4, 307–315.
- [9] H. Dai & X. Fang, On the distribution of coefficients of modular forms modulo  $p^j$ . *Proceedings of the AMS*, published electronically on October 18, 2016, <http://dx.doi.org/10.1090/proc/13323>
- [10] A. Granville & K. Soundararajan, The distribution of values of  $L(1, \chi_d)$ . *Geom. Funct. Anal.* 13 (2003), no. 5, 992–1028.
- [11] H. Halberstam & H.-E. Richert, Sieve methods. *London Mathematical Society Monographs*, No. 4. Academic Press, London-New York, (1974)
- [12] J.-L. Nicolas, I. Z. Ruzsa, & A. Sárközy, On the parity of additive representation functions. With an appendix in French by J.-P. Serre. *J. Number Theory* 73 (1998), no. 2, 292–317.
- [13] J.-L. Nicolas, Valeurs impaires de la fonction de partition  $p(n)$ . *Int. J. Number Theory* 2 (2006), no. 4, 469–487.
- [14] J.-L. Nicolas, Parité des valeurs de la fonction de partition  $p(n)$  et anatomie des entiers. *Anatomy of integers*, 97–113, CRM Proc. Lecture Notes, 46, Amer. Math. Soc., Providence, RI, 2008.
- [15] K. Ono, The web of Modularity: Arithmetic of the Coefficients of Modular forms and  $q$ -Series. *CBMS Regional Conference Series in Mathematics*, 102 (2004).
- [16] K. Ono & C. Skinner, Fourier Coefficients of Half-Integral Weight Modular Forms Modulo  $l$ . *Annals of Mathematics*, Second Series, Vol. 147, No. 2 (1998), pp. 453–470.
- [17] F. Zanello, On the number of odd values of the Klein  $j$ -function and the cubic partition function. *J. Number Theory* 151 (2015), 107–115.

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